IDENTIFICATION OF STATIONARY NOISE SOURCES BASED ON A FINITE ELEMENT ENHANCED FORMULATION

Rivânia H. Paulino de Romero[a] and **Fernando Alves Rochinha**[b]

[a] *Department of Mechanical Engineering, Federal University of Espirito Santo, UFES*
Vitória, ES, Brasil

[b] *Department of Mechanical Engineering, Federal University of Rio de Janeiro, UFRJ*
Rio de Janeiro, RJ, Brasil

faro@serv.com.ufrj.br

Abstract

This paper deals with discrete inverse problem in acoustics. It is assumed that the number of acoustic sources are located at known spatial positions and that the acoustic velocity is measured at a number of spatial positions in the radiated field. The situations will be restricted to steady states and due to the type of application that is being envisaged only internal acoustic problems. The Macroelement Post-Processing recovery technique is based on residuals of equilibrium equation and irrotationality conditions. The derivatives are recovered by solving local variational problems exploring special superconvergence behavior occurring in acoustic problems. This finite element enhanced formulation is explored in the context of estimating noise sources. A number of comprehensive simulation is presented in order to assess the main features of proposed formulation.

1 Introduction

Nowadays, numerical modelling plays a crucial role in engineering, specially whether complex systems are involved. It enhances the prediction capability when direct measurements are not accessible or numerical simulations are cheaper than carrying out experiments. It also allows the analysis of critical situations in which testing becomes prohibiting.

On the other hand, designing engineering systems using numerical simulations requires reliable models which, in turn, relies on the the available data and on assumptions involving modelling and discretization. The usual way of improving the situation is adjusting the model by means of comparing the model results with information obtained through experiments or field measurements, which is generically referred to as model validation.

One important issue involved on model validation is providing the modelling with reliable data as, for instance, the identification of noise sources in acoustic problems, which is the focus of the present work. Identifying the acoustic source enables the prediction of sound radiation by means of finite element models which might serv as a departure point for noise control design leading to the improvement of acoustic quality.

Source identification is normally achieved indirectly as direct measurements are often cumbersome. Therefore, the source reconstruction is phrased as an inverse problem [4] combining some experimental data, modelling and optimization strategies. A inverse formulation based on finite elements aiming at source identification is presented here.

Neglecting non-linear effects (convection, advection), which is justified by the low level of acoustic pressure p inducing very small perturbations around a steady uniform state in an inviscid fluid, the acoustic wave propagation is described by the Helmolhtz equation [3], e.g.

$$\nabla^2 p + k^2 p = 0 \tag{1}$$

with k is the so called *wave number* defined by

$$k = \frac{\omega}{c} \tag{2}$$

where ∇^2 stands for the Laplacian operator and ω and c are, respectively, the circular frequency and the material's speed of sound.

In order to have a complete description of the wave propagation phenomena, it is necessary to choose the appropriate boundary conditions, which in the present context are presented bellow

$$\nabla p \cdot \mathbf{n} = -ikc\rho A p \quad on \quad \Gamma_R \tag{3}$$

$$p = p_0 \quad on \quad \Gamma_p \tag{4}$$

where ρ is the mass density, $\mathbf{n}$ is the exterior unit normal vector, $i = \sqrt{-1}$ and A is the admittance coefficient. Moreover, the acoustic velocity vector is defined as

$$\mathbf{v} = \frac{-1}{iw\rho}\nabla p \tag{5}$$

In the identification situation addressed in the present article, the source is assumed to be stationary and placed over the portion of the boundary Γ_p. It is modelled by $p(x,t) = P_0(x)e^{iwt}$ with w known, therefore the identification problem is set as finding the spatial distribution $P_0(x)$ corresponding to the strength of the source. The identification formulation involves model output data which is obtained trough an enhanced finite element formulation, a crucial issue of the proposed methodology, outlined later. This finite element formulation explores superconvergence behavior of in special points which will provide extra accuracy for the identification problem. A similar idea is used in [7], where elastic parameters of a viscoelastic beam are to be identified.

The remainder of this paper is organized as follows. Section 2 presents the direct problem, in which the pressure field radiation is computed once the boundary conditions, geometry and material parameters are known. Its discretized counterpart is also introduced. Section 3 contains the inverse problem formulation together with a numerical algorithm to solve the resulting problem. Section 4 presents a numerical example that is used to assess the performance of the proposed formulation. Section 5 is devoted to final remarks.

2 Direct Problem: Weak formulation of the governing equation

In order to solve by means of the finite element method the interior acoustic problem introduced above, the following variational formulation based on the primal variable p is introduced.

PROBLEM P: Find $p \in V$ such that

$$\int_\Omega \nabla p . \nabla \bar{q}\, d\Omega - k^2 \int_\Omega p\, \bar{q}\, d\Omega + ik\rho c \int_{\Gamma_p} A p\, \bar{q}\, d\Gamma = 0 \tag{6}$$

$\forall q \in V$ where the function space of admissible pressure fields V is defined as

$$V = \{p \in H^1(\Omega) \quad | \quad q = 0 \quad on \quad \Gamma_p\} \tag{7}$$

where $H^1(\Omega)$ is the Sobolev space in which functions and theirs generalized derivatives are square integrable. Moreover, $\bar{q}$ stands for the complex conjugate of q, throughout the rest of the paper the bar will be omitted in order to use a simpler notation.

2.1 Finite element discretization

The Galerkin finite element approximation of the equation (6) is obtained by considering a finite dimensional spaces $V_h \subset V$, and then seeking an approximate solution to (6). Let S_h^k be quadrilateral or triangular C^0 lagrangian finite element subspace of $H^1(\Omega)$ of degree k. The discrete version of PROBLEM P in $V_h = S_h^k \cap V$ reads

PROBLEM P_h: Find $p_h \in V_h$, such that

$$\begin{aligned} \int_\Omega \nabla p_h . \nabla q_h d\Omega \quad & -k^2 \textstyle\int_\Omega p_h\, q_h d\Omega + \\ ik\rho c \textstyle\int_{\Gamma_p} A p_h\, q_h d\Gamma \quad & = 0 \quad \forall q_h \in V_h \end{aligned} \tag{8}$$

3 Source identification: the inverse problem

The source identification is phrased as an inverse problem in which the unknown strength distribution is sought. In this type of problem, this lack of information is somewhat compensated by measuring extra data over a portion of the domain. In the present work, the adopted strategy relies on obtaining the acoustic velocity field over Γ_R, for which the admittance coefficient is a priori known.

Thus, the identification problem is formulated as finding the distribution $P_0(x)$ on Γ_R which furnishes a minimum of the the least square objective function given by:

$$J(P_0) = \int_{\Gamma_M} (\mathbf{v}^M - \mathbf{v}_h^*(P_0))^2 d\Gamma \tag{9}$$

where $\mathbf{v}^M$ represents the measured velocity field and $\mathbf{v}^*(P_0)$ is the one obtained by solving the direct problem for an assumed value of P_0. Indeed, as the direct problem is addressed by the finite element method, the source is associate to a vector of nodal values corresponding to the discretization of P_0. For the sake of compactness, the same notation is adopted for the sought field and its discrete counterpart.

The minimization problem is subject to the following restrictions engendered by the equations corresponding to the direct problem:

$$\left.\begin{array}{rcl} \nabla^2 p_h + k^2 p_h & = & 0 \quad em \quad \Omega \\ p_h(\mathbf{x}) & = & P_0(\mathbf{x}) \quad x \in \Gamma_p \\ \nabla p_h.\mathbf{n} & = & -ikp_h \quad \mathbf{x} \in \Gamma_M \end{array}\right\} \tag{10}$$

In order to solve numerically the above minimization problem, it is necessary a reformulation involving the following Lagrangian:

$$\begin{array}{l} L(P_0, \lambda) = \displaystyle\int_{\Gamma_M} (\mathbf{v}^M - \mathbf{v}_h^*(P_0))^2 d\Gamma + \\ \int_\Omega \lambda.(\nabla^2 p_h + k^2 p_h) d\Omega \end{array} \tag{11}$$

where λ it is the lagrange multiplier responsible for incorporating the restriction constituted by the state equations. For obtaining the minimum of the functional L, the method of the gradient [2] will be used. The corresponding iterative algorithm is synthesized by its l-ésima iteration, through the expression:

$$P_0{}^{l+1} = P_0{}^l - \alpha_t L^{'}(P_0{}^l) \quad (l = 1, ..., n) \tag{12}$$

where the positive scalar α_t is the step size which is often decisive in the process of convergence of the iterative method and $L^{'}(P_0{}^l)$ represents the gradient of the lagrangian, which supplies the descent direction, computed in the point $P_0{}^l$.

3.1 Model Output $\mathbf{v_h}^*$: Macroelement Post-Processing Technique

Broadly speaking, the identification problem consists on minimizing the least-square difference between the measured velocity field and the corresponding model's output. Often, this problem is ill conditioned which entails on the results a high degree of sensitivity to small perturbations on the data. Therefore, an accurate approximation of $\mathbf{v_h}^*$ is required, which, in the present work, is provided by the post-processing technique described next.

Usually, after solving the Helmholtz equation, the velocity field is obtained by using a discrete version of (5). This direct approach leads to lower-order accuracy for the derivatives when compared to the primal variable p_h. This motivates the development of the often called recovery strategies, in which a more accurate approximation of the derivative variable is sought. Here, the macroelement post-processing technique proposed in [1] is applied to acoustic problems.

The basic idea of this method is to find a better approximation for the derivative (∇p) by solving on each macroelement, understood as the union of neighboring elements with common edges, a local variational problem involving the residuals of the balance equation, of the irrotationality condition and the constitutive relation at special superconvergence points [5]. To this end we suppose that the domain Ω is decomposed into macroelements not necessarily disjoints and define $MQ_h^s \subset L^2(\Omega)$ quadrilateral lagrangian finite element spaces of piecewise polynomial of degree s on each element and class C^0 in each macroelement but discontinuous on the macroelement boundaries.

Considering $Z_h = MQ_h^s \times MQ_h^s$, the following residual form for the problem of retrieving a better estimate for the velocity field is introduced PROBLEMS ME_h Given $p_h \in V_h$ solution of PROBLEM P_h, find $\mathbf{v}_h^* \in Z_h = MQ_h^s \times MQ_h^s$, such that

$$\begin{array}{l} \left(\mathbf{v}_h^* + \dfrac{i}{\omega\rho}\nabla p_h, \mathbf{u}_h\right)_h + \\ \delta_1 h^2 \left(\nabla \cdot \mathbf{v}_h^* + \dfrac{i\omega}{\rho c^2} p_h, \nabla \cdot \mathbf{u}_h\right) + \\ \delta_2 h^2 (\nabla \times \mathbf{v}_h^*, \nabla \times \mathbf{u}_h) = 0 \; \forall \; \mathbf{u_h} \in Z_h \end{array} \tag{13}$$

where $\mathbf{v}_h^*$ is the post-processed velocity and a more compact notation was adopted, in which $(.,.)$ represents the $L2$ product and $(.,.)_h$ denotes that this term is evaluated by suitable integration rule, taking into account the superconvergence of the derivatives. Besides, $\nabla.$ and $\nabla\times$ stand for the divergence and rotational operators. The scalars δ_1 and δ_2 are positive real parameters, that might be adjusted in order to have better convergence performance, and h is the mesh parameter. The above variational formulation is formed by three terms, namely: the first one is associate to the constitutive relationship between pressure and velocity evaluated in superconvergent points and the last two are derived from least-squares residuals of the continuity equation and the irrotationality condition, respectively.

3.2 Gradient Computation: the adjoint problem

The optimization algorithm (12) requires the evaluation of the Lagrangian's gradient at each iteration, which consists on the more complex step of the numerical procedure. There are different ways of obtaining this gradient detailed in [9]. Here, an adjoint formulation is adopted. It starts by introducing the the directional derivative $D_{\delta P_0} L(\mathbf{P_0})$ formally detailed bellow:

$$D_{\delta P_0} L(P_0) = \int_{\Gamma_M} (-2)\ (\mathbf{v}^M - \mathbf{v}_h^*) \cdot (\breve{\mathbf{v}}_h^*)\ d\Gamma + \int_\Omega \lambda . (\nabla^2 \breve{p}_h + k^2 \breve{p}_h)\ d\Omega \tag{14}$$

where $\breve{\mathbf{v}}$ e $\breve{\mathbf{p}}$,often referred to as sensitivities, are defined through

$$\breve{p}_h(x, P_0; \delta P_0) = \frac{d}{d\eta} p_h(x, P_0 + \eta \delta P_0)\bigg|_{\eta=0} \tag{15}$$

and

$$\breve{\mathbf{v}}_h^* = \frac{\partial \mathbf{v}_h^*}{\partial p_h} \breve{p}_h \tag{16}$$

where the operator $\dfrac{\partial \mathbf{v}_h^*}{\partial p_h}$ will be detailed later. As, from the computational standpoint, obtaining those sensibilities is very expensive, the following alternative is introduced. Applying integration by parts, equation (14) is rewritten and reads as:

$$D_{\delta P_0} L(P_0) = \int_{\Gamma_M} (-2)(\mathbf{v}^M - \mathbf{v}_h^*) \cdot \breve{\mathbf{v}}_\mathbf{h}^*\ d\Gamma + \int_\Gamma \lambda \cdot (\nabla \breve{p}_h \cdot \mathbf{n}) d\Gamma - \int_\Gamma (\nabla \lambda \cdot \mathbf{n}) \cdot \breve{p} d\Gamma + \int_\Omega (\nabla^2 \lambda + k^2 \lambda) \cdot \breve{p} d\Omega \tag{17}$$

Dividing the integrals over the boundary Γ through the partition: $\int_\Gamma = \int_{\Gamma_p} + \int_{\Gamma_M}$ yields

$$D_{\delta P_0} L(P_0) = \int_{\Gamma_M} (-2)(\mathbf{v}^M - \mathbf{v}_h^*) \cdot \breve{\mathbf{v}}_\mathbf{h}^*\ d\Gamma + \int_{\Gamma_p} \lambda \cdot (\nabla \breve{p}_h \cdot \mathbf{n}) d\Gamma + \int_{\Gamma_M} \lambda \cdot (\nabla \breve{p}_h \cdot \mathbf{n}) d\Gamma - \int_{\Gamma_p} (\nabla \lambda \cdot \mathbf{n}) \breve{p}_h d\Gamma - \int_{\Gamma_M} (\nabla \lambda \cdot \mathbf{n}) \cdot \breve{p}_h d\Gamma + \int_\Omega (\nabla^2 \lambda + k^2 \lambda) \cdot \breve{p}_h d\Omega \tag{18}$$

Introducing the adjoint problem defined below

$$\begin{cases} \nabla^2 \lambda + k^2 \lambda &= 0 \quad em \quad \Omega \\ \lambda &= 0 \quad \mathbf{x} \in \Gamma_f \\ \nabla \lambda \cdot \mathbf{n} &= (\mathbf{v}^M - \mathbf{v}_h^*) \cdot \left(-2 \dfrac{\partial \mathbf{v}_h^*}{\partial p_h} \right) \mathbf{x} \in \Gamma_M \end{cases} \tag{19}$$

and applying the boundary conditions of the problem, the directional derivative is thus given by

$$D_{\delta P_0} L(P_0) = - \int_{\Gamma_p} (\nabla \lambda \cdot \mathbf{n}) \cdot \breve{p}_h d\Gamma \tag{20}$$

Therefore, the gradient computation is reduced, for each direction δP_0, to an integration over the boundary Γ_p involving the pressure sensitivity, which for that region is trivially computed, and the lagrange multiplier obtained through the solution of (19).

For solving (19), one needs the velocity sensitivity over Γ_M which is obtained with the aid of the post-processing equation, given rise to the following local auxiliary problem

$$\mathbf{b}_h\left(\frac{\partial \mathbf{v}_\mathbf{h}^*}{\partial p_h}, \mathbf{u}_h\right) = \frac{\partial F(p_h, \mathbf{u}_h)}{\partial p_h} \qquad \forall \mathbf{u_h} \tag{21}$$

where $\mathbf{b}_h$ stands for the finite element operator defined from the post-processing and $F(p_h, \mathbf{u_h})$ is given by:

$$F(p_h, \mathbf{u_h}) = -\left(\frac{i}{w\rho} \nabla p_h, u_h \right) - \delta_1 h^2 \left(\frac{iw}{\rho c^2} p_h, \nabla \cdot u_h \right) \tag{22}$$

4 Duct 1D: model problem for source identification

The performance of the proposed approach is assessed through a problem involving a simple geometry that reproduces in essence the

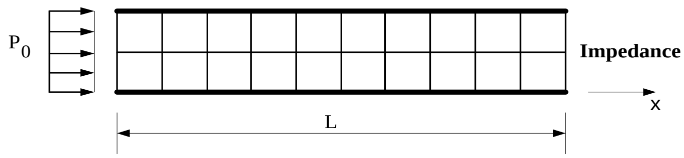

Figure 1: Duct geometry, boundary conditions and typical mesh

$\delta's$	p_r	p_i	$\dfrac{\lVert q^{l+1}-q^l\rVert}{\lVert q^l-q^{l-1}\rVert}$
$0,001$	$2,000$	$1,14\times10^{-5}$	$0,84$
$0,1$	$2,004$	$-7,52\times10^{-4}$	$0,89$
$0,25$	$2,006$	$-1,13\times10^{-2}$	$0,91$
$1,0$	$2,008$	$-1,59\times10^{-2}$	$0,94$
10	$2,009$	$-1,69\times10^{-2}$	$0,95$
100	$2,009$	$-1,71\times10^{-2}$	$0,94$
1000	$2,009$	$-1,71\times10^{-2}$	$0,87$

Table 1: Variations of the parameter δ for an estimative of error

sound propagation in an one-dimensional duct, see Figure (1) for details. In the present test, experiments are substituted by simulations, so the measured data is obtained by solving the direct problem for a prescribed source (constant over Γ_p with real and imaginary components equal to, respectively, 2 and 0), which for the identification procedure is assumed not known. In the duct's right extremity, $x = L$, sensors aiming at measuring the velocity are placed coinciding with the three nodal points of the mesh presented in Figure (1), for which $h = 0,1m$.

The iterative procedure is illustrated by the scheme (2).

The first analysis which was carried out refers to varying the post-processing parameters $\delta's$ using the same initial estimate (real component of the pressure (p_r) equals to 6 and the imaginary one (p_i) equals to -3), with the purpose of evaluating the convergence rate and precision of the results by using the proposed enhanced finite element formulation. Those results are summarized in Table 1 for different values of $\delta's$ with $\delta = \delta_1 = \delta_2$. Fixing the optimization step $\alpha = 1$ a quasi linear convergence rate was achieved.

A summary of the convergence history is provided by figures 3, 4, and 5.

Aiming at testing the robustness of the proposed formulation, several different values for

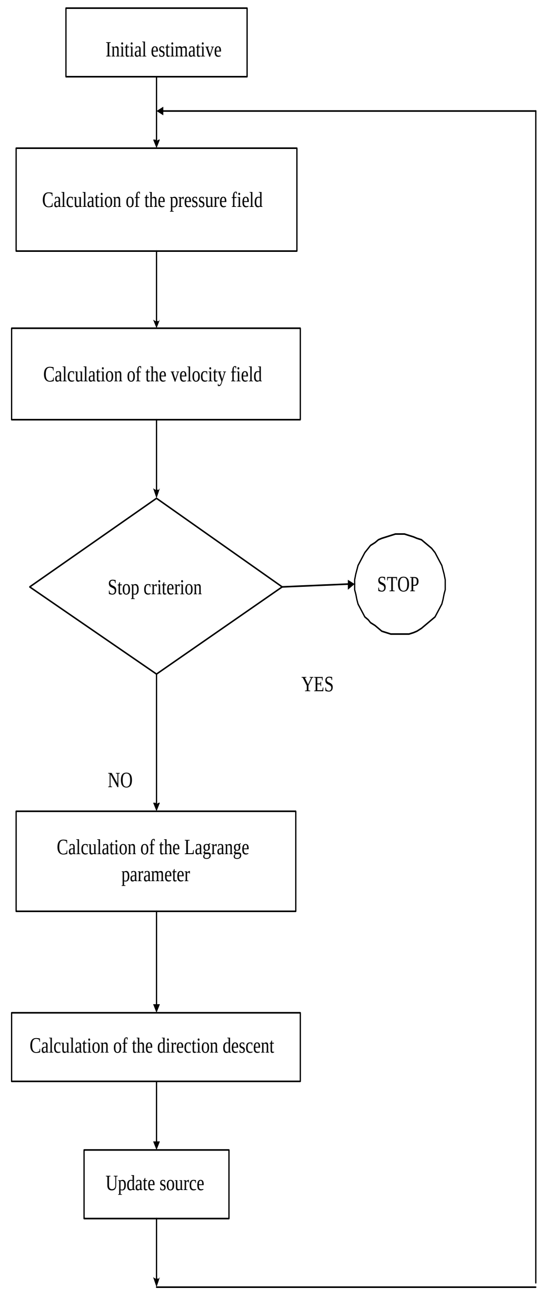

Figure 2: Schematic view of the proposed algorithm

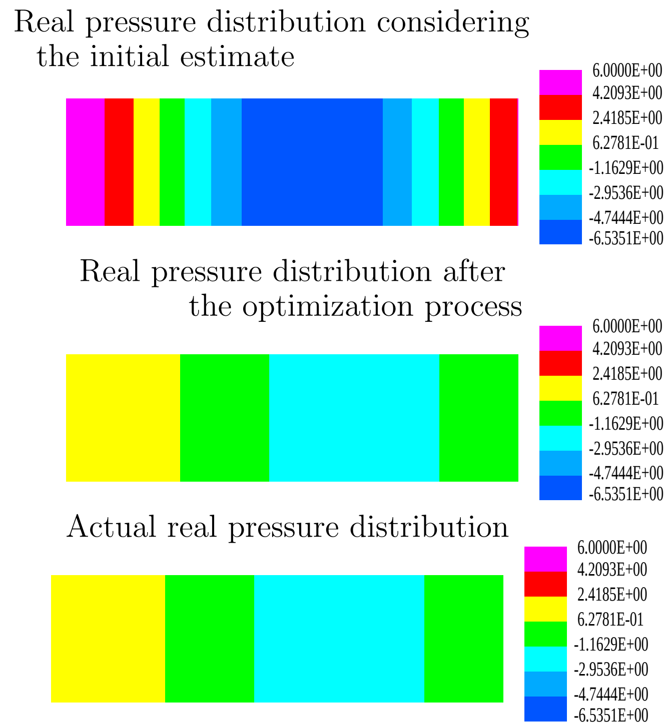

Figure 3: Real pressure distribution along the duct

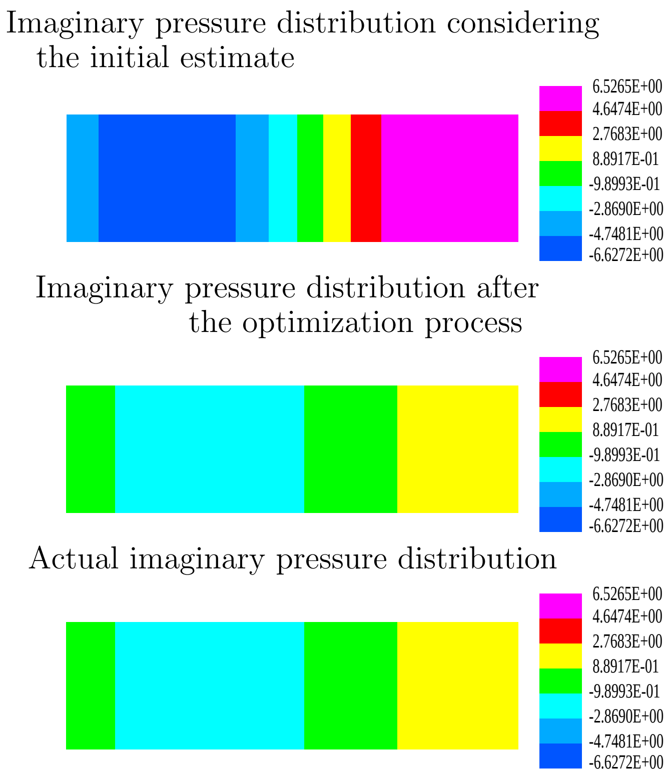

Figure 4: Imaginary pressure distribution along the duct

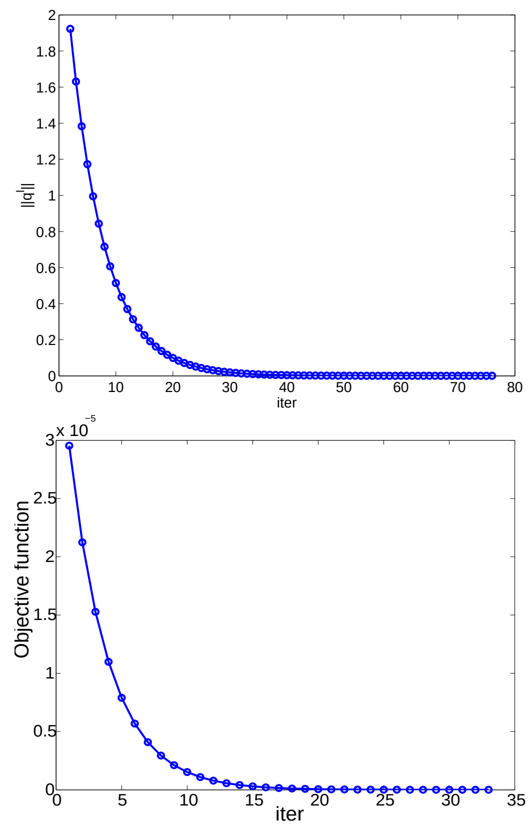

Figure 5: History of the optimization process, $h = 0.1m$, $k = 5$, $\delta = 0.001$

the initial estimative were taken. Tables 2 and 3 furnish the final results. It is important to note that good estimations of the noise source were obtained independently of the initial estimate, confirming the robustness of the proposed formulation.

The convergence was reached, for all cases, with a relatively low number of iterations. Figure (6) contains the evolution of the objective function along the iterative process.

In order to stress the difficulties involving the estimation of the noise source, the initial estimates were assumed not constant over Γ_p. In all situations presented in Table 4, the iterative procedure ended leading to incorrect values of the source strength. The obtained results represent local minima of the formulation which surely requires a refinement on the optimization algorithm.

Motivated by the situation described in the previous paragraph, further tests were carried out in order to ensure the enhancement provided by the post-processing to the source estimation problem.

The inverse problem was formulated without the participation of the post-processing given

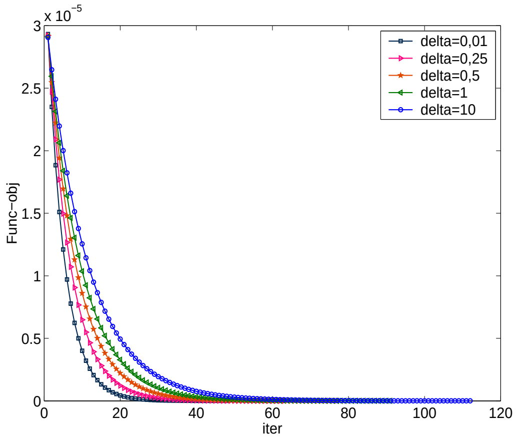

Figure 6: History of the optimization process, $h = 0.1m$, $k = 5$

p_r^i	p_i^i	p_r^f	p_i^f
$-0,001$	10	$2,006$	$-1,14 \times 10^{-2}$
$-2,0$	$1,0$	$2,006$	$-1,14 \times 10^{-2}$
$-10,0$	$5,0$	$2,006$	$-1,14 \times 10^{-2}$
$6,0$	$-3,0$	2.006	$-1,13 \times 10^{-2}$
$17,0$	$-10,0$	$2,006$	$-1,13 \times 10^{-2}$
$-150,0$	$-70,0$	$2,006$	$-1,13 \times 10^{-2}$
$200,0$	$100,0$	$2,006$	$-1,14 \times 10^{-2}$
$-1000,0$	$-1000,0$	$2,006$	$-1,13 \times 10^{-2}$

Table 2: Variations of the initial estimative of the pressure field, $\delta = 0,25$

by the equation 13, in other words, to the parameters $\delta's$ are attributed values zero and the term $(.)_h$ was fully integrated, not taking into account the superconvergence behavior. In brief, an $L_2(\Omega)$ regularization in adopted.

When comparing the two formulations, in other words, a formulation based on the post-processing and the other based on the regularization in $L_2(\Omega)$, for the same initial guesses ,it is noticed through the results shown in the tables 3 and 5 that the former furnishes better results. The computational cost was also an advantage, as the needed number of iterations was less for the post-processing formulation. Several other tests were accomplished and it can be said that the proposed formulation is a promising tool for the target applications.

p_r^i	p_i^i	p_r^f	p_{imag}^f
$-0,001$	10	$2,000$	$-1,69 \times 10^{-5}$
$-2,0$	$1,0$	$2,000$	$-4,38 \times 10^{-6}$
$-10,0$	$5,0$	$2,000$	$-1,23 \times 10^{-6}$
$17,0$	$-10,0$	$2,000$	$3,43 \times 10^{-6}$
$-150,0$	$-70,0$	$2,000$	$1,89 \times 10^{-5}$
$200,0$	$100,0$	$2,000$	$-2,14 \times 10^{-5}$
$-1000,0$	$-1000,0$	$2,000$	$-1,33 \times 10^{-5}$

Table 3: Variations of the initial estimative of the pressure field, $\delta = 0,001$

$p_{real}^{inicial}$	$p_{imag}^{inicial}$	p_{real}^{final}	p_{imag}^{final}
$0,01$	0.00	1.96	$-2,55 \times 10^{-2}$
$0,10$	0.05	2.04	$2,49 \times 10^{-2}$
$0,01$	0.00	1.96	$-2,55 \times 10^{-2}$
$0,01$	$-0,1$	$1,99$	$-5,00 \times 10^{-2}$
$0,02$	-0.05	2.00	$-2,15 \times 10^{-6}$
$0,03$	0.00	2.01	$4,99 \times 10^{-2}$
$-1,00$	$0,50$	$1,80$	9.99×10^{-2}
$-0,10$	0.05	2.70	$-3,50 \times 10^{-1}$
$-2,00$	1.00	0.80	$6,00 \times 10^{-1}$
$3,00$	$1,00$	$1,00$	$-1,00 \times 10^{0}$
$4,00$	2.00	2.00	$-1,92 \times 10^{-6}$
$5,00$	3.00	3.00	$-2,55 \times 10^{-2}$
$10,00$	$5,00$	$17,00$	$7,50 \times 10^{0}$
$-20,00$	-10.00	-13.00	$-7,50 \times 10^{0}$
$10,00$	5.00	17.00	$7,50 \times 10^{0}$

Table 4: Variations of the initial estimative of the pressure field, $\delta = 0,001$

$p_{real}^{inicial}$	$p_{imag}^{inicial}$	p_{real}^{final}	p_{imag}^{final}
$-0,001$	10	$1,975$	$-2,59 \times 10^{-1}$
$-2,0$	$1,0$	$1,975$	$-2,59 \times 10^{-1}$
$-10,0$	$5,0$	$1,975$	$-2,59 \times 10^{-1}$
$6,0$	$-3,0$	$1,975$	$-2,59 \times 10^{-1}$
$17,0$	$-10,0$	$1,975$	$-2,59 \times 10^{-1}$
$-150,0$	$-70,0$	$1,975$	$-2,59 \times 10^{-1}$
$200,0$	$100,0$	$1,975$	$-2,59 \times 10^{-1}$
$-1000,0$	$-1000,0$	$1,975$	$-2,59 \times 10^{-1}$

Table 5: Variations of the estimative intial of the pressure field, alternative formulation (regularization in $L_2(\Omega)$)

5 Final Remarks

The more important contribution of the present work relies on the use of an finite element enhanced formulation for obtaining the acoustic velocity field, which, in turn, provides the basis for an inverse problem aiming at the estimation of noise sources.

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